A new class of Ornstein transformations with singular spectrum

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A NEW CLASS OF ORNSTEIN TRANSFORMATIONS WITH SINGULAR SPECTRUM
UNE NOUVELLE CLASSE DE TRANSFORMATIONS D'ORNSTEIN A SPECTRE SINGULIER

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Abstract. It is shown that for any family of probability measures in Ornstein type constructions the corresponding transformation has almost surely a singular spectrum. This is a new generalization of Bourgain's theorem [7], the same result is proved for Rudolph's construction [20].

RÉSUMÉ. On montre que pour toute famille de mesures de probabilités dans la construction d'Ornstein, les transformations résultantes ont un spectre presque sûrement singulier. On obtient ainsi une nouvelle généralisation d'un théoréme dû à Bourgain [7]. Un résultat similaire est obtenu pour les transformations de Rudolph [20].

1. Introduction

In this note we investigate the spectral analysis of a generalized class of Ornstein transformations. There are several generalizations of Ornstein transformations. Here we are concerned with arbitrary product probability space associated to random construction of the family of rank one transformations. Namely, in the Ornstein's construction, the probability space is equipped with the infinite product of uniform probability measures on some finite subsets of $\mathbb{Z}$. Here, the probability space is equipped with the infinite product of probability measures $(\xi_m)_{m\in\mathbb{N}}$ on a family $(X_m)_{m\in\mathbb{N}}$ of finite subsets of $\mathbb{Z}$. We establish that for any choice of the family $(\xi_m)_{m\in\mathbb{N}}$ the associated Ornstein transformations has almost surely singular spectrum.

Let us recall that Ornstein introduced these transformations in 1967 in [15] and proved that the mixing property occurs almost surely. Until 1991, these transformations which have simple spectrum appeared as a candidate for an affirmative

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answer to Banach's well-known problem whether a dynamical system $(X, \mathcal{B}, \mu)$ may have simple Lebesgue spectrum. But, in 1991, J. Bourgain in [7], using Riesz products techniques, proved that Ornstein transformations have almost surely singular spectrum. Subsequently, I. Klemes [18], I. Klemes & K. Reinhold [19] obtain that the spectrum of the mixing subclass of staircase transformations of T. Adams [5] and T. Adams & N. Friedman [6] have singular spectrum. They conjectured that rank one transformations allways have singular spectrum.

In this paper, using the techniques of J. Bourgain generalized in [1], we extend Bourgain's theorem to the generalized Ornstein transformations associated to a large family of random constructions.

Firstly, we shall recall some basic facts from spectral theory. A nice account can be found in the appendix of [16]. We shall assume that the reader is familiar with the method of cutting and stacking for constructing rank one transformations.

Given $T : (X, \mathcal{B}, \mu) \mapsto (X, \mathcal{B}, \mu)$ a measure preserving invertible transformation and denoting by $U_T f$ the operator $U_T f(x) = f(T^{-1}x)$ on $L^2(X, \mathcal{B}, \mu))$, recall that to any $f \in L^2(X)$ there corresponds a positive measure σ_f on $\mathbb{T}$, the unit circle, defined by $\hat{\sigma}_f(n) =< U_T^n f, f >$.

Definition 1.1. The maximal spectral type of T is the equivalence class of Borel measures σ on $\mathbb{T}$ (under the equivalence relation $\mu_1 \sim \mu_2$ if and only if $\mu_1 << \mu_2$ and $\mu_2 << \mu_1$), such that $\sigma_f << \sigma$ for all $f \in L^2(X)$ and if ν is another measure for which $\sigma_f << \nu$ for all $f \in L^2(X)$ then $\sigma << \nu$.

There exists a Borel measure $\sigma = \sigma_f$ for some $f \in L^2(X)$, such that σ is in the equivalence class defining the maximal spectral type of T. By abuse of notation, we will call this measure the maximal spectral type measure. The reduced maximal type σ_0 is the maximal spectral type of U_T on $L_0^2(X) \stackrel{\text{def}}{=} \{f \in L^2(X) \ : \ \int f d\mu = 0\}$. The spectrum of T is said to be discrete (resp. continuous, resp. singular, resp. absolutely continuous , resp. Lebesgue) if σ_0 is discrete (resp. continuous, resp. singular, resp. absolutely continuous with respect to the Lebesgue measure or equivalent to the Lebesgue measure). We write

$$Z(h) \stackrel{\text{def}}{=} \overline{\text{span}\{U_T^n h, n \in \mathbb{Z}\}}.$$

T is said to have simple spectrum, if there exists $h \in L^2(X)$ such that

$$Z(h) = L^2(X).$$

2. Rank One Transformation by Construction

Using the cutting and stacking method described in [12], [13], one defines inductively a family of measure preserving transformations, called rank one transformations, as follows

Let B_0 be the unit interval equipped with the Lebesgue measure. At stage one we divide B_0 into p_0 equal parts, add spacers and form a stack of height h_1 in the usual fashion. At the k^{th} stage we divide the stack obtained at the $(k-1)^{th}$ stage into p_{k-1} equal columns, add spacers and obtain a new stack of height h_k. If during the k^{th} stage of our construction the number of spacers put above the j^{th} column of the $(k-1)^{th}$ stack is $a_j^{(k-1)}$, $0 \leq a_j^{(k-1)} < \infty$, $1 \leq j \leq p_k$, then we have

$$\begin{aligned} h_k &= p_{k-1}h_{k-1} + \sum_{j=1}^{p_{k-1}} a_j^{(k-1)}, \ \forall k \geq 1, \\ h_0 &= 1. \end{aligned}$$

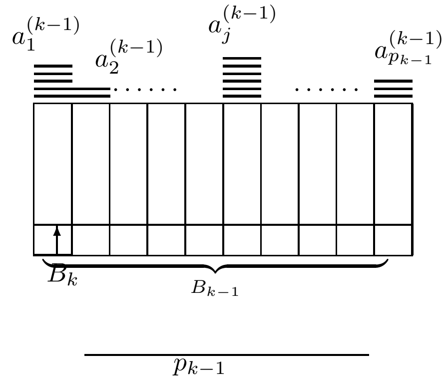

Figure 1: k^{th}–tower.

Proceeding in this way we get a rank one transformation T on a certain measure space $(X, \mathcal{B}, \nu)$ which may be finite or $\sigma-$finite depending on the number of spacers added.
The construction of any rank one transformation thus needs two parameters $(p_k)_{k=0}^{\infty}$ (cutting parameter) and $((a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}$ (spacers parameter). We put

$$T \stackrel{def}{=} T_{(p_k,(a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}}$$

In [7],[9] and [19] it is proved that up to some discret measure, the spectral type of this transformation is given by

$$d\sigma = W^* \lim \prod_{k=1}^{n} |P_k|^2 \, d\lambda. \tag{2.1}$$

where $P_k(z) = \dfrac{1}{\sqrt{p_k}} \left(\displaystyle\sum_{j=0}^{p_k-1} z^{-(jh_k + \sum_{i=1}^{j} a_i^{(k)})} \right)$

λ denotes the normalized Lebesgue measure on torus $\mathbb{T}$.

W^* denotes weak convergence on the space of bounded Borel measures on $\mathbb{T}$.

The polynomials P_k appear naturally from the induction relation between the bases B_k. Indeed

$$B_k = B_{k+1} \cup T^{h_k+s_k(1)}B_{k+1} \cup \ldots \cup T^{(p_k-1)h_k+s_k(p_k-1)}B_{k+1},$$

$$\nu(B_k) = p_k \nu(B_{k+1}),$$

where $s_k(n) = a_1^{(k)} + \ldots + a_n^{(k)}$ and $s_k(0) = 0$.
Put

$$f_k = \frac{1}{\sqrt{\nu(B_k)}}\chi_{B_k},$$

that is the indicator function of the kth-base normalized in the L^2-norm. So

$$f_k = \ P_k(U_T)f_{k+1},$$

where $U_T \ : \ L^2(X) \longrightarrow L^2(X)$ is defined by $U_T(f)(x) = f(T^{-1}x)$. Iterating this relation, we have

$$d\sigma_k = |P_k|^2 \, d\sigma_{k+1} = \ldots = \prod_{j=0}^{m-1} |P_{k+j}|^2 \, d\sigma_{k+m},$$

Where σ_p is the spectral measure of f_p, $p \geq 0$.

3. Generalized Ornstein's Class of Transformations

In Ornstein's construction, the p_k's are rapidly increasing, and the number of spacers, $a_i^{(k)}$, $1 \leq i \leq p_k - 1$, are chosen randomly. This may be organized in differently ways as pointed by J. Bourgain in [7]. Here, We suppose given (t_k), (p_k) a sequences of positive integers and (ξ_k) a sequence of probability measure such that the support of each ξ_k is a subset of $X_k = \{-\frac{t_k}{2}, \cdots, \frac{t_k}{2}\}$. We choose now independently, according to ξ_k the numbers $(x_{k,i})_{i=1}^{p_k-1}$, and x_{k,p_k} is chosen deterministically in $\mathbb{N}$. We put, for $1 \leq i \leq p_k$,

$$a_i^{(k)} = t_k + x_{k,i} - x_{k,i-1}, \text{ with } x_{k,0} = 0.$$

It follows

$$h_{k+1} = p_k(h_k + t_k) + x_{k,p_k}.$$

So the deterministic sequences of positive integers $(p_k)_{k=0}^{\infty}$, $(t_k)_{k=0}^{\infty}$ and $(x_{k,p_k})_{k=0}^{\infty}$ determine completely the sequence of heights $(h_k)_{k=0}^{\infty}$. The total measure of the resulting measure space is finite if

$$\sum_{k=0}^{\infty} \frac{t_k}{h_k} + \sum_{k=0}^{\infty} \frac{x_{k,p_k}}{p_k h_k} < \infty. \tag{3.1}$$

We will assume that this requirement is satisfied.
We thus have a probability space of Ornstein transformations $\Omega = \prod_{l=0}^{\infty} X_l^{p_l-1}$ equipped with the natural probability measure $\mathbb{P} \stackrel{\text{def}}{=} \otimes_{l=1}^{\infty} P_l$, where $P_l \stackrel{\text{def}}{=} \otimes_{j=1}^{p_l-1}\xi_l$; ξ_l is the probability measure on X_l. We denote this space by $(\Omega, \mathcal{A}, \mathbb{P})$. So $x_{k,i}$, $1 \leq$

$i \leq p_k - 1$, is the projection from Ω onto the i^{th} co-ordinate space of $\Omega_k \stackrel{\text{def}}{=} X_k^{p_k-1}$, $1 \leq i \leq p_k - 1$. Naturally each point $\omega = (\omega_k = (x_{k,i}(\omega))_{i=1}^{p_k-1})_{k=0}^{\infty}$ in Ω defines the spacers and therefore a rank one transformation $T_{\omega,x}$, where $x = (x_{k,p_k})$.
The definition above gives a more general definition of random construction due to Ornstein. In the particular case of Ornstein's transformations constructed in [15], $t_k = h_{k-1}$, ξ_k is uniform distribution and $p_k >> h_{k-1}$.
We recall that Ornstein in [15] proved that there exist a sequence $(p_k, x_{k,p_k})_{k\in\mathbb{N}}$ such that, $T_{\omega,x}$ is almost surely mixing. Later in [17], Prikhod'ko obtain the same result for some special choice of the sequence of the distribution (ξ_m) and recently, using the idea of D. Creutz and C. E. Silva [10] one can extend this result to a large class of the family of the probability measure associated to Ornstein construction.
In our general construction, according to (2.1) the spectral type of each T_ω , up to a discrete measure, is given by

$$\sigma_{T_\omega} = \sigma_{\chi_{B_0}}^{(\omega)} = \sigma^{(\omega)} = W^* \lim \prod_{l=1}^{N} \frac{1}{p_l} \left| \sum_{p=0}^{p_l-1} z^{p(h_l+t_l)+x_{l,p}} \right|^2 d\lambda.$$

With the above notation, we state our main result

Theorem 3.1. *For every choice of $(p_k), (t_k), (x_{k,p_k})$ and for any family of probability measures ξ_m on finite subset X_m of $\mathbb{Z}$, $m \in \mathbb{N}^*$. The associated generalized Ornstein transformations has almost surely singular spectrum. i.e.*

$$\mathbb{P}\{\omega \ : \ \sigma^{(\omega)} \perp \lambda\} = 1.$$

Where $\mathbb{P} \stackrel{\text{def}}{=} \otimes_{l=0}^{\infty} \otimes_{j=1}^{p_l-1} \xi_l$; is the probability measure on $\Omega = \prod_{l=0}^{\infty} X_l^{p_l-1}$, X_l is finite subset of $\mathbb{Z}$.

Before proceeding to the proof, we remark that it is an easy exercise to see that the spectrum of Ornstein's transformation is always singular if the cutting parameter p_k is bounded. In fact, Klemes-Reinhold proved moreover that if $\displaystyle\sum_{k=0}^{\infty} \frac{1}{{p_k}^2} = \infty$ then the associated rank one transformation is singular. Henceforth, we assume that the series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{{p_k}^2}$ converges.

We shall adapt Bourgain's proof. For that, we need a local version of the singularity criterion used by Bourgain. Let F be a Borel set then with the above notations, we will state local singularity criterion in the following form

Theorem 3.2. (Local Singularity Criterion (LSC)) *The following are equivalent*

(i) $\sigma_F \perp \lambda$, , *where $\sigma_F = \chi_F.d\sigma$, χ_F is a indicator function of F.*

(ii) $\displaystyle\int_F \prod_{l=1}^{n} |P_l(z)|\, d\lambda \xrightarrow[n\to\infty]{} 0.$

(iii) $\inf\{\int_F \prod_{l=1}^{k} |P_{n_l}(z)|\, d\lambda,\ k \in \mathbb{N},\ n_1 < n_2 < \ldots < n_k\} = 0.$

One can adapt the proof of theorem 4.3 in [19], or in [1], [14], in the more general setting.
Now, using Lebesgue's dominated convergence theorem and the LSC , we obtain

Proposition 3.3. *The following are equivalent*
(i) $\sigma_F^{(\omega)} \perp \lambda \qquad \mathbb{P}\ a.s.$
(ii) $\int_F \prod_{l=1}^{n} |P_l(z)|\, d\lambda d\mathbb{P} \xrightarrow[n\to\infty]{} 0.$
(iii) $\inf\{\int \int_F \prod_{l=1}^{k} |P_{n_l}(z)|\, d\lambda d\mathbb{P},\ k \in \mathbb{N},\ n_1 < n_2 < \ldots < n_k\} = 0.$

Fix some subsequence $\mathcal{N} = \{n_1 < n_2 < \ldots < n_k\}$, $k \in \mathbb{N}$, $m > n_k$ and put

$$Q\,(z) \quad = \quad \prod_{i=1}^{k} |P_{n_i}(z)|.$$

Following [7] (see also [18] or in the more general setting [3]), we have.

Lemma 3.4.

$$\int_F Q\,|P_m|\ \ d\lambda\ \le \frac{1}{2}\left(\int_F Q d\lambda + \int_F Q\,|P_m|^2\, d\lambda\right) - \frac{1}{8}\left(\int_F Q \left||P_m|^2 - 1\right| d\lambda\right)^2.$$

Now, we assume that F is closed set, it follows

Lemma 3.5. $\limsup_{m\to\infty} \int_F Q\,|P_m(z)|^2\, d\lambda(z) \le \int_F Q\ d\lambda(z).$

Proof : Observe that the sequence of probability measures $|P_m(z)|^2\, d\lambda(z)$ converges weakly to the Lebsegue measure. Then the lemma follows from the classical portmanteau theorem [1] and the proof is complete. ■

From the lemmas 3.4 and 3.5 we get the following

Lemma 3.6.

$$\liminf \iint_F Q\,|P_m|\, d\lambda d\mathbb{P} \le \iint_F Q d\lambda d\mathbb{P} - \frac{1}{8}\left(\limsup \iint_F Q \left||P_m|^2 - 1\right| d\lambda d\mathbb{P}\right)^2.$$

Clearly, we need to estimate the quantity

$$\int \int_F Q \left||P_m(z)|^2 - 1\right| d\lambda(z) d\mathbb{P}. \tag{3.2}$$

For that, following Bourgain we shall prove the following

[1] see for example [11]. We note that the space Ω is equiped with the standard product topology.

Proposition 3.7. *There exists an absolute constant $K > 0$ such that*

$$\limsup \int\int_F Q\left||P_m|^2 - 1\right| d\lambda d\mathbb{P} \geq K\Big(\int\int_F Q d\lambda d\mathbb{P} - \liminf \int\int_F Q(z)\phi_m(z) d\lambda d\mathbb{P}\Big)^2,$$

where $\phi_m(z) = \left|\sum_{p=-\frac{t_m}{2}}^{p=\frac{t_m}{2}} \xi_m(p) z^p\right|^2$, $z \in \mathbb{T}$

We shall give the proof of proposition 3.7 in the following section.

4. Khintchine-Bonami inequality

Fix $z \in \mathbb{T}$ and $m \in \mathbb{N}^*$. Define τ and $(\tau_p)_{p=1}^{p_m-1}$ by :

$$\begin{aligned} \tau : \mathbb{Z} &\longrightarrow \mathbb{T} \\ s &\longmapsto z^s. \end{aligned}$$

τ_p is given by $\tau_p = \tau \circ x_{m,p}$, $x_{m,p}$ is the p^{th} projection on $\Omega_m = X_m^{p_m-1}$. So

$$|P_m(z)|^2 - 1 = \sum\nolimits_{p \neq q} a_{pq}\ \tau_p(\omega)\overline{\tau_q}(\omega). \quad \text{where } a_{pq} = \frac{z^{(p-q)(h_m+t_m)}}{p_m},$$

The random variables $(\tau_p)_{p=1}^{p_m-1}$ are independent. Put

$$\tau_p^\circ = \tau_p - \int \tau_p\ d\mathbb{P},\ p = 1, \cdots, p_m - 1. \tag{4.1}$$

and write

$$\begin{aligned} &\sum a_{pq}\ \tau_p\ \overline{\tau_q} = \\ &\Big(\sum a_{pq}\Big)\left|\int \tau_1\right|^2 + \sum a_{pq}\left(\left(\int \overline{\tau_1}\right)\ \tau_p^\circ + \left(\int \tau_1\right)\ \overline{\tau_q^\circ}\right) + \sum a_{pq}\tau_p^\circ\ \overline{\tau_q^\circ}. \end{aligned} \tag{4.2}$$

Now, using the same arguments as J. Bourgain, let us consider a random sign $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_{p_m-1}\} \in \{-1, 1\}^{p_m-1}$, and the probability space

$$Z_m = \Omega_m \times \{-1, 1\}^{p_m-1},\ \text{where } \Omega_m = \left\{-\frac{t_m}{2}, ..., \frac{t_m}{2}\right\}^{p_m-1}.$$

Taking the conditional expectation of the following quantity

$$\sum a_{pq}\left(\int \tau_p^\circ\ \overline{\tau_1} + \int \overline{\tau_q^\circ}\ \tau_1\right) + \sum a_{pq}\tau_p^\circ\ \ \overline{\tau_q^\circ}$$

with respect to the $\sigma-$algebra $\mathcal{B}_\varepsilon$ given by the cylindres sets $A(I, x)$ where $I \subset \{1, \ldots, p_m - 1\}$, $x \in \Omega_m$ and

$$A(I, x) = \prod_{i \in I}\{x_i\} \times \left\{-\frac{t_m}{2}, ..., \frac{t_m}{2}\right\}^{|I^c|} \times \{1\}^{|I|} \times \{-1\}^{|I^c|}.$$

(I corresponds to $\varepsilon_i = 1$, $\forall i \in I$ and $\varepsilon_i = -1$, $\forall i \notin I$). In other words, taking conditional expectation with respect to the random variables τ_p for which $\varepsilon_p = 1$, one finds the following polynomial expression in ε of degree 2

$$\sum a_{pq}\left(\frac{1+\varepsilon_p}{2}\int \overline{\tau_1}\ \ \tau_p^\circ + \frac{1+\varepsilon_q}{2}\int \tau_1\ \ \overline{\tau_q^\circ}\right) + \sum a_{pq}\frac{1+\varepsilon_p}{2}\frac{1+\varepsilon_q}{2}\tau_p^\circ\ \ \overline{\tau_q^\circ} \tag{4.3}$$

So

$$\begin{aligned}\int \Big||P_m(z)|^2-1|\Big|\, d\mathbb{P} &= \int\int \mathbb{E}(\Big||P_m(z)|^2-1\Big|_{|\mathcal{B}_\varepsilon})\; d\mathbb{P}d\varepsilon \\ &\geq \int\int \Big|\mathbb{E}(|P_m(z)|^2-1_{|\mathcal{B}_\varepsilon})\;\Big|\, d\mathbb{P}d\varepsilon.\end{aligned} \tag{4.4}$$

It follows, by the Khintchine-Bonami inequality, [2] [8], that there exists a positive constant K such that

$$\begin{aligned}&\int\int \Big|\mathbb{E}(|P_m(z)|^2-1_{|\mathcal{B}_\varepsilon}]\;\Big|\, d\mathbb{P}d\varepsilon \\ &\geq K\int (\int \Big|\mathbb{E}(|P_m(z)|^2-1_{|\mathcal{B}_\varepsilon}]\;\Big|^2\, d\varepsilon)^{\frac{1}{2}} d\mathbb{P} \\ &= K\int \left(\sum_{p\neq q}\Big|a_{pq}(z)\tau_p^\circ(z)\overline{\tau_q^\circ(z)}\Big|^2\right)^{\frac{1}{2}} d\mathbb{P}.\end{aligned} \tag{4.5}$$

But all these random variables are bounded by 2. Hence

$$\begin{aligned}&\int \Big||P_m(z)|^2-1\Big|\, d\mathbb{P} \\ &\geq K'\int \frac{1}{p_m^2}\sum \Big|\tau_p^\circ(z)\overline{\tau_q^\circ(z)}\Big|^2\, d\mathbb{P} \\ &= K'\frac{1}{p_m^2}\sum\left(\int \big|\tau_p^\circ(z)\big|^2\, d\mathbb{P}\right)^2 \\ &= K'\frac{(p_m-1)(p_m-2)}{{p_m}^2}\left(\int |\tau_1^\circ(z)|^2\, d\mathbb{P}\right)^2.\end{aligned} \tag{4.6}$$

Since

$$\begin{aligned}&\int |\tau_1^\circ(z)|^2 d\mathbb{P} = var(\tau_1(z)) \\ &= 1-\left|\sum_{s=-\frac{t_m}{2}}^{\frac{t_m}{2}} \xi_m(s)z^s\right|^2.\end{aligned} \tag{4.7}$$

Now, combined (4.6) with (4.7) to obtain

$$\begin{aligned}&\int \Big||P_m(z)|^2-1\Big|\, d\mathbb{P} \\ &\geq K'\frac{(p_m-1)(p_m-2)}{{p_m}^2}\left(1-\left|\sum_{s=-\frac{t_m}{2}}^{\frac{t_m}{2}} \xi_m(s)z^s\right|^2\right)^2\end{aligned} \tag{4.8}$$

Finally, Multiply (4.8) by

$$\int \prod_{j\in\mathcal{N}} |P_j(z)|\, d\mathbb{P}. \tag{4.9}$$

[2]One can extend easily this inequality to bounded sequences of independent real random variables, with vanishing expectation.

Using the independence of (4.9) and $|1-|P_m(z)|^2|$. Integrating over F with respect to the Lebesgue measure to get

$$\begin{aligned} &\int_\Omega \int_F Q \left||P_m(z)|^2 - 1\right| d\lambda d\mathbb{P} \\ &\geq K'(\int_\Omega \int_F Q(1-\phi_m(z))^2 d\lambda d\mathbb{P}) \end{aligned} \tag{4.10}$$

where $\phi_m(z) = \left|\sum_{s=-\frac{t_m}{2}}^{\frac{t_m}{2}} \xi_m(s) z^s\right|^2$. Apply Cauchy-Schwarz inequality to obtain

$$\begin{aligned} \int_\Omega \int_F Q(1-\phi_m(z)) d\lambda d\mathbb{P} &\leq \left(\int_\Omega \int_F Q d\lambda d\mathbb{P}\right)^{\frac{1}{2}} \left(\int_\Omega \int_F Q(1-\phi_m(z))^2 d\lambda d\mathbb{P}\right)^{\frac{1}{2}} \\ &\leq \left(\int_\Omega \int_F Q(1-\phi_m(z))^2 d\lambda d\mathbb{P}\right)^{\frac{1}{2}} \end{aligned} \tag{4.11}$$

Combined (4.10) and (4.11) and take liminf to finish the proof of the proposition 3.7. ■

Now, passing to a subsequence we may assume that ϕ_m converge weakly in $L^2(\lambda)$ to some function ϕ in $L^2(\lambda)$. Then,

$$\widehat{\phi}(n) = \lim_{m \longrightarrow \infty} \widehat{\phi_m}(n) \geq 0. \text{ for any } n \in \mathbb{Z},$$

and

$$\sum_n \widehat{\phi}(n) \leq 1.$$

Hence, the Fourier series of ϕ converge absolutely and we may assume

$$\phi(z) = \sum_n \widehat{\phi}(n) z^n,$$

In particular ϕ is a continuous function. We deduce that the set $\{\phi(z) = 1\}$ is either the torus or a finite subgroup of the torus.

Remark 4.1. It is any easy exercise to see that if the set $\{\phi = 1\}$ is not a null set with respect to Lebesgue measure then, for any $z \in \mathbb{T}$,

$$\phi(z) = 1$$
$$\text{and } \max_{s \in X_m} \xi_m(s) \xrightarrow[m \to \infty]{} 1.$$

We shall, now, prove, our main result in the following sections.

5. On the Ornstein probability space for which $\underline{\lim} \max_{s \in X_m} \xi_m(s) < 1$

In this section, we assume that $\underline{\lim} \max_{s \in X_m} \xi_m(s) < 1$. So, we may choose ϕ the weak limite of subsequence of ϕ_m so that $\widehat{\phi}(0) < 1$ and $\{\phi = 1\}$ is a finite. Let $\varepsilon > 0$, put

$$F_\varepsilon \stackrel{def}{=} \{z \in \mathbb{T} \ : \ 1 - \phi(z) \geq \varepsilon\}.$$

We get easily that F_ε is a closed set and we have also the following proposition

Proposition 5.1. *There exists an absolute constant $K > 0$ such that*

$$\underline{\lim} \iint_{F_\varepsilon} Q \left| |P_m(z)|^2 - 1 \right| d\lambda \mathbb{P} \geq K\varepsilon^2 \Big(\iint_{F_\varepsilon} Q d\lambda \mathbb{P} \Big)^2.$$

Proof : Apply the proposition 3.7 to get that there exists an constant $K > 0$ for which we have

$$\begin{aligned}
&\liminf \iint_{F_\varepsilon} Q \left| |P_m(z)|^2 - 1 \right| d\lambda \mathbb{P} \\
\geq \;& K \left(\iint_F Q d\lambda d\mathbb{P} - \overline{\lim} \iint_F Q(z) \left| \sum_{p=-\frac{t_m}{2}}^{p=\frac{t_m}{2}} \xi_m(p) z^p \right|^2 \lambda d\mathbb{P} \right)^2 \\
\geq \;& K \Big(\iint_{F_\varepsilon} Q \left(1 - \phi(z)\right) \lambda d\mathbb{P} \Big)^2 \\
\geq \;& K\varepsilon^2 \Big(\iint_{F_\varepsilon} Q \lambda d\mathbb{P} \Big)^2.
\end{aligned}$$

The proof of the proposition is complete. ■

Proof of the theorem 3.1.in the case of $\underline{\lim} \max_{s \in X_m} \xi_m(s) < 1$

First, for fixed $\varepsilon > 0$, let us choose the good subsequence $\mathcal{N} \stackrel{def}{=} \{n_k, k \geq 0\}$. Observe that from the propositions 3.6. and 5.1. one can write

$$\overline{\lim} \int\int_{F_\varepsilon} Q \left| P_m(z) \right| d\lambda(z) d\mathbb{P} \leq \int\int_{F_\varepsilon} Q - \frac{1}{8} K^2 \varepsilon^4 \left(\int\int_{F_\varepsilon} Q d\lambda d\mathbb{P} \right)^4,$$

and from this last inequality we shall construct $\mathcal{N}$. In fact, suppose we have chosen the k first elements of the subsequence $\mathcal{N}$. We wish to define the $(k+1)^{\text{th}}$ element. Let $m > n_k$ such that

$$\int\int_{F_\varepsilon} Q \left| P_m(z) \right| d\lambda(z) d\mathbb{P} \leq \int\int_{F_\varepsilon} Q d\lambda d\mathbb{P} - \frac{1}{8} K^2 \varepsilon^4 \left(\int\int_{F_\varepsilon} Q d\lambda d\mathbb{P} \right)^4,$$

and put $n_{k+1}{=}m$. It follows that the elements of the subsequence $\mathcal{N}$ verify

$$\int\int_{F_\varepsilon} \prod_{i=1}^{k+1} |P_{n_i}(z)| d\lambda d\mathbb{P} \leq \int\int_{F_\varepsilon} \prod_{i=1}^{k} |P_{n_i}(z)| d\lambda d\mathbb{P} - \frac{1}{8} K^2 \varepsilon^4 \left(\int\int_{F_\varepsilon} \prod_{i=1}^{k} |P_{n_i}(z)| d\lambda d\mathbb{P} \right)^4.$$

We deduce that the sequence $(\int\int_{F_\varepsilon} \prod_{i=1}^{k} |P_{n_i}(z)| d\lambda \mathbb{P})_{k \geq 1}$ is decreasing and converges to the limit l_ε which verifies

$$l_\varepsilon \leq l_\varepsilon - \frac{1}{8} K^2 \varepsilon^4 l_\varepsilon^4,$$

and this implies that $l_\varepsilon = 0$. Hence, $\sigma_{F_\varepsilon}^{(\omega)}$ is singular. But,

$$\bigsqcup_{\varepsilon > 0, \varepsilon \in \mathbb{Q}} \{1 - \phi \geq \varepsilon\} = \{1 - \phi \neq 0\},$$

and by our assumption ($\underline{\lim} \max_{s \in X_m} \xi_m(s) < 1$) we choose ϕ such that $\{1-\phi(z) = 0\}$ is a null set with respect to the Lebsegue measure. This complete the proof of theorem 3.1. when $\underline{\lim} \max_{s \in X_m} \xi_m(s) < 1$. $\square$

6. On the Ornstein probability space for which $\underline{\lim} \max_{s \in X_m} \xi_m(s) = 1$

Using the same ideas as in the previous section, we have the following

Lemma 6.1. $\displaystyle \limsup_{m \longrightarrow \infty} \iint ||P_m|^2 - 1| d\lambda d\mathbb{P} \geq \iint Q d\lambda d\mathbb{P}.$

Proof : We have

$$\iint Q \left||P_m|^2 - 1\right| d\lambda d\mathbb{P} \geq \int \left| \int Q \left(|P_m|^2 - 1\right) d\mathbb{P} \right| d\lambda, \tag{6.1}$$

But, from (4.2)

$$\begin{aligned} \int \left(|P_m|^2 - 1\right) d\mathbb{P} = {} & 2\mathrm{Re}\Big\{ \left(\mathrm{G_{p_m}(z^{h_m+t_m})}\right) \left(\int \tau_1 \mathrm{d}\mathbb{P} \right) \Big\} \\ & + \left| F_{p_m}(z^{h_m+t_m}) - \frac{p_m - 1}{p_m} \right| \phi_m(z). \end{aligned} \tag{6.2}$$

Where, F_p and G_p is define, for any $p \in \mathbb{N}^*$, by

$$\begin{aligned} F_p(z) &= \left| \frac{1}{\sqrt{p}} \sum_{k=1}^{p-1} z^k \right|^2, \\ G_p(z) &= \frac{1}{p} \sum_{k=1}^{p-1} z^k. \end{aligned}$$

Re(z) is a real part of the complex number z. Combined (6.1) and (6.2) to obtain

$$\begin{aligned} \iint Q \left||P_m|^2 - 1\right| d\lambda d\mathbb{P} \quad \geq \quad & \iint Q \left(\left| F_{p_m}(z^{h_m+t_m}) - \frac{p_m - 1}{p_m} \right| \phi_m(z) \right) d\lambda \mathbb{P} - \\ & 2 \iint Q \left| G_{p_m}(z^{h_m+t_m}) \right| \left| \int \tau_1 d\mathbb{P} \right| d\lambda d\mathbb{P}. \end{aligned} \tag{6.3}$$

But, on one hand, we have

$$\begin{aligned} \int Q \left| G_{p_m}(z^{h_m+t_m}) \right| \left| \int \tau_1 d\mathbb{P} \right| d\lambda \quad & \leq \quad \left(\int \left| G_{p_m}(z^{h_m+t_m}) \right|^2 d\lambda \right)^{\frac{1}{2}} \left(\int Q^2 d\lambda \right)^{\frac{1}{2}} \\ & \leq \quad \left(\int \left| G_{p_m}(z^{h_m+t_m}) \right|^2 d\lambda \right)^{\frac{1}{2}} = \frac{1}{\sqrt{p_m}} \xrightarrow[m \to \infty]{} 0. \end{aligned}$$

On the other hand, since $|X_m| \leq t_m$, $\sum_{k \in X_m} \left(\xi_m\{k\}\right)^2 \xrightarrow[m \to \infty]{} 1$ and for any $f \in L^1$, we have

$$\widehat{f_{(m)}}(n) = \begin{cases} 0 & \text{if } n \text{ is not divisible by } m \\ \widehat{f}\left(\dfrac{n}{m}\right) & \text{otherwise} \end{cases}$$

Where $f_{(m)}(z) = f(z^m)$, we get that $\left|F_{p_m}(z^{h_m+t_m}) - \frac{p_m-1}{p_m}\right| \left|\sum_{k\in X_m} \xi_m(k) z^k\right|^2 d\lambda$ converge to $K.\lambda$, with $K \geq 1$. In fact

$$\begin{aligned}
&\int \left|F_{p_m}(z^{h_m+t_m}) - \frac{p_m-1}{p_m}\right| \left|\sum_{k\in X_m} \xi_m(k) z^k\right|^2 d\lambda \\
= &\sum_{k\in X_m} (\xi_m\{k\})^2 \int \left|F_{p_m}(z^{h_m+t_m}) - \frac{p_m-1}{p_m}\right| d\lambda \\
\geq &\sum_{k\in X_m} (\xi_m\{k\})^2 \int \left(F_{p_m}(z^{h_m+t_m}) - \frac{p_m-1}{p_m}\right) z^{h_m+t_m} d\lambda \\
= &\sum_{k\in X_m} (\xi_m\{k\})^2 \left(\frac{p_m-2}{p_m}\right) \xrightarrow[m\to\infty]{} 1.
\end{aligned}$$

and the proposition follows from (6.1). ■

Proof of the theorem 3.1.in the case of $\underline{\lim} \max_{s\in X_m} \xi_m(s) = 1$

As in the case of $\underline{\lim} \max_{s\in X_m} \xi_m(s) < 1$, we use the lemma (6.1) to etablish that

$$\lim_{n\longrightarrow\infty} \int \prod_{k=1}^{n} |P_k(z)| d\lambda d\mathbb{P} = 0.$$

and the proof of the theorem 3.1. is complete.

Remark 6.2. We note that Rudolph construction in [20] is strictly included in the theory of generalized random Ornstein construction.

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